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Speculative Note on Kinematics

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Abstract

A probabilistic kinematics is described. The point of departure is the introduction of probability distributions for measurements of length and time. For macroscopic systems, the usual Lorentz transformations remain valid. Fundamental parameters of length and time, which measure the dispersions of the probability distributions, are postulated to be Lorentz invariants. Comments are included concerning the establishment of coordinate systems. The kinematics itself does not indicate how the probabilistic character of space and time measurements is to be incorporated into dynamical calculations, and additional prescriptions are necessary. In order both to investigate dynamical applications and to set a limit on the values of the fundamental parameters, calculations are undertaken of the kinematical corrections to the potential and kinetic energies for the shift between the hydrogenic $2s$ and $2p$ levels.

1. *Introduction*

Speculations on a way in which the kinematics that underlies the laws of physics may be modified to become probabilistic in character are described. The conceptual basis of the speculations lies in the intuitive idea that it should be impossible to speak of physical events occurring at a point in space-time. A more practical argument for avoiding the concept of a point in space-time is the appearance of difficulties in various theories on account of singularities that appear in the equations. An attractive objective is to work with a kinematics that precludes point singularities in principle. Toward this end the introduction of a fundamental length into the laws of physics has often been discussed.

In the present speculations, strict validity of the Lorentz transformations is presumed only for macroscopic systems. The point of departure is the introduction of probability distributions for measurements of length and time. The parameters that measure the dispersions of the distributions are postulated to be Lorentz invariants.

2. *Kinematical Postulates*

It is postulated that the Lorentz transformations are to be replaced by the probability laws described here. For the most part, the discussion is carried out in terms of a single spatial variable, x, and the variable t, where t is the speed of light times the time. The observations described in the statement of the postulates are stated in strictly classical terms and possible complications of quantum origin are ignored. This is in keeping with a guiding philosophy that the kinematical and dynamical aspects of the problems are to be kept separate. Simultaneous observations of a pair of complementary variables is not implied. In view of the probabilistic character of measurements of spatial coordinates and the time, numerous questions must arise as to how one can establish coordinate systems. Some comments concerning these questions are given in Appendix A.

Consider a set of N observers all of whom observe a single event and record its place and time of occurrence as x and t. All N observers are in the same Lorentz reference system. It is postulated that the number of observers who find the event occurring in the spatial interval x to $x + \Delta x$ and in the time interval t to $t + \Delta t$ is $N_{xt}\Delta x\Delta t$, with

$$N_{xt}\Delta x\Delta t = Nf(x, x_0, L)\Delta x f(t, t_0, T)\Delta t \tag{2.1}$$

Here x_0 and t_0 denote space and time positions about which the events are expected to cluster. The parameters L and T are measures of the spreads of the space and time observations, respectively, from the centers of the probability distributions. Conventionally, we would say that all observers observe x_0 and t_0 only. In order to avoid gross inconsistencies with current agreement between experiment and theory we must take L and T to be of the order of dimensions associated with elementary particles, i.e., $L < 10^{-13}$ cm, approximately. We also suspect that, in the units employed, $L \approx T$ because the speed of light is the only fundamental kinematical constant. If we prefer, an alternative expression of the postulate may be given in terms of repeated observations of a set of events by a single observer in which the events are repeatedly set up with similar initial conditions. The functions $f(x, x_0, L)$ and $f(t, t_0, T)$ evidently must have the property

$$\int_{-\infty}^{\infty} dx\, f(x, x_0, L) = \int_{-\infty}^{\infty} dt\, f(t, t_0, T) = 1 \tag{2.2}$$

With the introduction of these probability distributions for observations of space and time positions, the concept of a *point* should no longer occur in a formalism. The nature of the idea is that, in applications, a point in space-time is replaced by an ensemble constructed in accordance with the forms postulated for the probability distributions.

Next, we consider N observers in a second Lorentz reference system which travels along the $+x$ axis with speed v relative to the first system. Observations made by observers in the second system are denoted by primes. The

number of observers who find the event occurring in the spatial interval x' to $x' + \Delta x'$ and in the time interval t' to $t' + \Delta t'$ is $N'_{xt}\Delta x'\Delta t'$, with

$$N'_{xt}\Delta x'\Delta t' = Nf(x', x'_0, L)\Delta x' f(t', t'_0, T)\Delta t' \tag{2.3}$$

The parameters L and T are to be invariant, as is the functional form of the probability distribution. Equations of the form (2.2) also hold for the primed variables.

The centers of the distributions in the two systems are connected by the Lorentz transformations

$$\begin{aligned} x'_0 &= \gamma(x_0 - vt_0) \\ t'_0 &= \gamma(t_0 - vx_0) \end{aligned} \tag{2.4}$$

where $\gamma = (1 - v^2)^{-1/2}$. In the units employed, v is the ratio of the relative speed of the two coordinate systems to the speed of light. In view of the probabilistic nature of the postulates, the speed v is no longer well defined, a consideration that will be discussed in a later section. It is evident that the probabilistic transformations go over into the usual Lorentz transformations for macroscopic dimensions, provided L and T are of a size characteristic of elementary particles and the distribution function f is sharply peaked about its center.

Finally, it is to be noted that the group property of the Lorentz transformations will not be lost. In this connection, it is not meaningful to discuss any direct relationship between the arguments of the probability distributions in different Lorentz systems, i.e., no equations directly relate x and t to x' and t'.

3. *Form of the Probability Distribution*

The probability distribution for the observation of space or time coordinates is expected to have a well-defined peak, which occurs at x_0 or t_0, and is expected to be symmetrical about the peak, in which case the distribution should depend on $(x - x_0)^2$ or $(t - t_0)^2$. These expectations are in accordance with establishment of the kinematics independently of dynamical considerations and do not envision a situation analogous to that in general relativity in which the geometry itself is determined by dynamical factors. If the latter point of view were adopted, the form of the probability distribution and, perhaps, the magnitudes of L and T would be influenced by dynamical factors.

A more restrictive consideration follows from the recognition that the postulate, Eq. (2.1), could have been stated in terms of the measurement of intervals of space-time between two events rather than in terms of the coordinates of a single event. In fact, since there must be some way in which the origin of the space-time coordinate system is defined, the postulate implicitly assumes that a probability distribution of similar form applies to the difference in coordinates between the event in question and the event

which defines the origin; the probability distribution for $t = x_a + x_b$ is to be of the same form as the probability distributions for x_a and x_b. Given that the probability distributions for x_a and x_b are of the forms $f(x_a, x_{a0}, L)$ and $f(x_b, x_{b0}, L)$, respectively, the number of observations of r that lie between r and $r + \Delta r$ is given by the convolution integral

$$N\Delta r \int_{-\infty}^{\infty} dx_a f(x_a, x_{a0}, L)\, f(r - x_a, x_{b0}, L) = N\Delta r g(r, r_0, L') \qquad (3.1)$$

where N is again the number of observations, so that

$$N \int_{-\infty}^{\infty} dr\, g(r, r_0, L') = N \qquad (3.2)$$

We demand that the distribution in r be of the same form as the distributions in x_a and x_b, hence, $g(r, r_0, L') = f(r, r_0, L')$. We further expect that the centers of the distributions will be connected according to $r_0 = x_{a0} + x_{b0}$. The parameters which measure the spreads of the probability distributions are expected to obey the inequality $L' > L$.

The normal of Gaussian distribution has the expected properties. If we identify

$$f(x_a, x_{a0}, L) = (2\pi)^{-1/2} L^{-1} \exp[-(x_a - x_{a0})^2/2L^2] \qquad (3.3)$$

then

$$g(r, r_0, L') = (2\pi)^{-1/2} L'^{-1} \exp[-(r - r_0)^2/2L'^2] \qquad (3.4)$$

with $r_0 = x_{a0} + x_{b0}$ and $L' = (2)^{1/2} L$. The spread parameters are identified as the standard deviations of the Gaussian distributions. A similar discussion may be given for the time variable. The restrictions that have been imposed do not uniquely demand the Gaussian distribution. For example, the Cauchy distribution $f(x, x_0, L) \propto L/[L^2 + (x - x_0)^2]$, which is familiar as the form of a resonance curve, is in accord with the expectations (Zehna, 1970). (The moments of the Cauchy distribution do not exist, which would make its use in some computations unsatisfactory.)

4. *Velocity*

Because of its importance in connection with the Lorentz transformations and because of its use in some subsequent considerations, the probability distribution for a component of velocity will be discussed. The x component of velocity is defined as $v = S/H$, where S is an interval of distance along the x axis and H is an interval of time. In accordance with the discussion in the preceding section, the probability distributions for S and H may be denoted $f(S, S_0, L')$ and $f(H, H_0, T')$, respectively. We also define $v_0 = S_0/H_0$. The

number of observers who find the x component of velocity in the interval v to $v + \Delta v$ is ($L' = \sqrt{2}L$, $T' = \sqrt{2}T$)

$$N_v \Delta v = N \Delta v \int_{-\infty}^{\infty} dH\,|H|\, f(H, H_0, T') f(S = Hv, S_0, L') \tag{4.1}$$

Gaussian distributions for $f(S, S_0, L')$ and $f(H, H_0, T')$ are inserted to give

$$\begin{aligned} N_v/N = {} & \pi^{-1}(v^2 T/L + L/T)^{-1} \exp[(-H_0{}^2/4)(1/T^2 + v_0{}^2/L^2)] \\ & + 2^{-1}\pi^{-1/2} H_0 (L^2 + v v_0 T^2)(L^2 + v^2 T^2)^{-3/2} \\ & \times \operatorname{erf}[(2LT)^{-1} H_0 (L^2 + v v_0 T^2)(L^2 + v^2 T^2)^{-1/2}] \\ & \times \exp[-4^{-1} H_0{}^2 (L^2 + v^2 T^2)^{-1} (v - v_0)^2] \end{aligned} \tag{4.2}$$

A Gaussian approximation to this distribution becomes valid if the velocity measurement is made over a mean time, H_0, large compared with the time parameter T and if the velocities of interest satisfy $|v - v_0| \ll |v_0|$. The distribution is then

$$N_v/N = (2\pi)^{-1/2} s^{-1} \exp[-(v - v_0)^2/2s^2] \tag{4.3}$$

with

$$s = 2^{1/2} |T_0|^{-1} (L^2 + v_0{}^2 T^2)^{1/2}$$

The interpretation that may be given to these considerations is that a Lorentz reference system may be defined with precision only if the observations needed to define the speed of the reference system are made over a time interval that is large compared with the time parameter T. As the knowledge improves of a velocity component with which a Lorentz reference system moves relative to another system, the knowledge is sacrificed of the corresponding component of position at which the velocity measurement was made; the product of the mean time over which the set of velocity measurements is made and the standard deviation of the Gaussian approximation to the velocity distribution cannot be less than L'.

5. *Relationship between L and T*

A time interval may be defined in terms of an interval of distance and a standard velocity. According to this prescription we may define the interval of time, H, as $H = 2S/c$, where S is a distance interval and c is the speed of light, which is taken to be unity (in our units) by definition. The measurement procedure is as follows: At time t_a a light pulse leaves spatial point $\mathbf{r}_a$. A reflector is located at spatial point $\mathbf{r}_b$. The reflected light pulse arrives back at spatial point $\mathbf{r}_a$ at time t_b. The time interval is $H = t_b - t_a$. The procedure measures H, not t_a and t_b individually. The interval of distance is $S = |\mathbf{r}_b - \mathbf{r}_a|$. Evidently, the measurement of the time interval involves the determination

of the three components of each of two spatial position, i.e., in terms of the measured quantities

$$S = [(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2]^{1/2} \tag{5.1}$$

The measurements of the various components of position yield probability distributions in accord with previous postulates. The probability distribution for H may then be worked out if we assume Gaussian distributions with standard deviations of L for each of the position components. The relationship between L and T is obtained if we consider the "asymptotic" case in which the time interval is sufficiently large that the spatial coordinates of interest satisfy, for example,

$$|(x_b - x_{b0}) - (x_a - x_{a0})| \ll |x_{b0} - x_{a0}| \tag{5.2}$$

Then, the probability distribution for S is given, with good approximation, as a Gaussian with standard deviation L'. A Gaussian probability distribution for a time interval should have a standard deviation of T', according to our previous postulates. Hence, we conclude that $L = T$.

It is inherent in this prescription for time interval measurements that the origin of a probability distribution for the time is assigned entirely to the measurements of position components and no consideration is given to a distribution of values for the speed of light. If a time standard is defined according to the prescription given here, the speed of light is not measured *per se*, but other velocities may be measured using the time standard that has been established by use of light pulses.

6. *Dynamical Considerations*

The foregoing considerations define the kinematics. Unlike special relativity, experiments that might test the kinematics, and that are strictly kinematical in nature, do not suggest themselves. For magnitudes of L and T that may be expected, departures from conventional theory should only occur in experiments conducted on a microscopic scale, and strictly kinematical experiments on that scale are difficult to envision.

Any test of the speculations necessarily seems to involve a dynamical application of the kinematics. It is to be emphasized that the manner in which the probabilistic nature of the kinematics is to be introduced into a dynamical calculation is not indicated by the kinematics itself, but additional prescriptions are necessary. Accordingly, there may be some element of truth in the kinematical speculations that becomes invalidated by an incorrect dynamical prescription.

In carrying through an example of a possible dynamical calculation, we try to obey the philosophy that the probabilistic nature of the kinematics is to be introduced in a classical sense and the quantum phenomena are to be superimposed in a conventional manner. This is in contrast to any attempt to "explain" the probabilistic quantum effects in terms of the probabilistic nature of the kinematics (which is regarded as unsound in principle) or,

alternatively, any attempt to weave together the probabilistic features of the kinematics and quantum mechanics in some relatively complicated manner. Other than a preference for simplicity, there is no argument against the latter type of procedure.

Quantum electrodynamics probably provides the most exacting tests of current theoretical physics. In order both to investigate a possible procedure for a dynamical application of the kinematics and to estimate an upper limit on the parameter L that does not lead to a contradiction with one of the more precisely verified applications of extant physical laws, we will compute a correction to the Lamb shift on the hydrogenic $2s$ state that might be attributed to the probabilistic nature of the kinematics. The computation is commenced for a neutral Yukawa type field (Yukawa, 1935) but is soon specialized to the case of zero mass for the field quanta to yield a modification of the coulomb potential.

The classical field amplitude, $U(\mathbf{r})$, is produced by a source distribution $p_1(\mathbf{r}')$. This field interacts with a second charge distribution, $p_2(\mathbf{r})$. The interaction potential is computed and used in a simple, conventional calculation of the energy of the hydrogenic $2s$ state. For the interaction of two point charges, we would conventionally take the two charge distributions, p_1 and p_2, to be delta functions. This has the connotation that, in the conventional formulation of the problem, each charge can be associated with a particular point in space. The probabilistic nature of the kinematics is introduced by replacement of the delta functions by probability distributions that represent the probability for observing each charge with particular values for the spatial coordinates. Although the effect is the same as the assumption of smeared-out charge distributions, the conceptual basis is different. The probability distributions are chosen to be Gaussian in form, although it has been noted that a uniqueness argument for this form is lacking.

The expression for $U(\mathbf{r})$ is

$$U(\mathbf{r}) = -4\pi \int G(\mathbf{r}, \mathbf{r}')\, p_1(\mathbf{r}')\, dv' \tag{6.1}$$

In order that $U(\mathbf{r})$ satisfy the classical field equation $(\nabla^2 - k^2)U(\mathbf{r}) = -4\pi p_1(\mathbf{r})$ we have, for the Green's function,

$$G(\mathbf{r}, \mathbf{r}') = -(4\pi)^{-1} \exp(-k\,|\mathbf{r} - \mathbf{r}'|)/|\mathbf{r} - \mathbf{r}'| \tag{6.2}$$

The interaction of the two charge distributions is

$$W = -4\pi \iint G(\mathbf{r}, \mathbf{r}')\, p_1(\mathbf{r}')\, p_2(\mathbf{r})\, dv\, dv' \tag{6.3}$$

The probability distributions are introduced according to $p_1 = q_1 f_1(\mathbf{r}', \mathbf{r}_1, L)$ and $p_2 = q_2 f_2(\mathbf{r}, \mathbf{r}_2, L)$, with

$$f_2(\mathbf{r}, \mathbf{r}_2, L) = (2\pi)^{-3/2} L^{-3} \exp[-(\mathbf{r} - \mathbf{r}_2)^2/2L^2] \tag{6.4}$$

and a similar expression for $f_1(\mathbf{r}', \mathbf{r}_1, L)$. The constants q_1 and q_2 are the values of the two interacting charges. The centers of the two probability distributions are denoted as $\mathbf{r}_1$ and $\mathbf{r}_2$.

New variables are introduced as $\mathbf{R} = \mathbf{r} - \mathbf{r}_2$, $\mathbf{R}' = \mathbf{r}' - \mathbf{r}_1$, and $\mathbf{R}_0 = \mathbf{r}_2 - \mathbf{r}_1$; then $\mathbf{S} = \mathbf{R} + \mathbf{R}_0$ and $\mathbf{S}' = \mathbf{R}' - \mathbf{S}$. The integration over dS' is carried out to give

$$W(R_0) = 2^{-3/2}\pi^{-1/2}q_1q_2R_0^{-1}L^{-1}\exp(-R_0^2/2L^2)\int_0^\infty dS\exp(-S^2/L^2)$$
$$\times[\exp(SR_0/L^2) - \exp(-SR_0/L^2)][\exp(L^2k^2/2 - Sk + S^2/2L^2)$$
$$\times \operatorname{erfc}(Lk/\sqrt{2} - S/\sqrt{2}L) - \exp(L^2k^2/2 + Sk + S^2/2L^2)$$
$$\times \operatorname{erfc}(Lk/\sqrt{2} + S/\sqrt{2}L)] \tag{6.5}$$

The complementary error function is $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$. We now specialize to the photon interaction by putting $k = 0$ to obtain

$$W(R_0) = (2\pi)^{-1/2}q_1q_2R_0^{-1}L^{-1}\exp(-R_0^2/2L^2)\int_0^\infty dS\exp(-S^2/2L^2)$$
$$\times[\exp(SR_0/L^2) - \exp(-SR_0/L^2)]\ \operatorname{erf}(S/\sqrt{2}L) \tag{6.6}$$

The integration may be carried out for the two limiting cases of $r = 0$ and $r \to \infty$ to obtain

$$\begin{aligned} W &= \pi^{-1/2}q_1q_2/L \qquad \text{for } r = 0 \\ W &= q_1q_2/R_0 \qquad \text{for } r \to \infty \end{aligned} \tag{6.7}$$

The latter result is the expected coulomb interaction for point charges separated by the distance R_0 while the former result (essentially the self-interaction) verifies that the interaction does not diverge for finite L.

The calculation need only be made to the lowest-order correction terms in the parameter L/a_0, where a_0 is the first Bohr radius. In the computation of the energy of the $2s$ level, we need only consider the term in the hydrogenic wave function of the form $\exp(-Zr/2a_0)$. We have then to compute

$$\langle 2s' | W | 2s' \rangle = 2^{-1}Z^3a_0^{-3}\int dv\exp(-Zr/a_0)\,W(r) \tag{6.8}$$

where the terminology $2s'$ is adopted to signify that the irrelevant part of the hydrogenic wave function is not included in the formulas. In the expression for W, given in Eq. (6.6), the variable R_0 is now replaced by r. The integration over r may be carried through. In some terms the integration over dS may also be carried through, and terms are arranged in order to extract the contribution that would come from the usual coulomb interaction between two point charges. Terms of order higher than $(L/a_0)^2$ are dropped. We find

$$\langle 2s' | W | 2s' \rangle = -2^{-1}Z^2e^2a_0^{-1} + 2^{-1}Z^4e^2a_0^{-3}L^2(K + \tfrac{1}{2}) \tag{6.9}$$

with

$$K = 2\int_0^\infty dw\,w\operatorname{erfc}(w) = \tfrac{1}{2} \tag{6.10}$$

The contribution that would come from the coulomb interaction of two point charges is recognized as $\langle 2s' | W_c | 2s' \rangle = -Z^2e^2/2a_0$.

The result for the kinematical shift on the $2s$ state is then

$$\langle 2s|W|2s\rangle - \langle 2s|W_c|2s\rangle = 2^{-1}Z^4e^2a_0^{-3}L^2 \tag{6.11}$$

For convenience this may be expressed as: shift = $3.3 \times 10^9 Z^4(L^2/a_0^2)$ MHz. The agreement between theory and experiment for this shift in hydrogen is about 0.1 MHz, so the parameter L cannot be larger than about 0.3 fermi without causing a serious discrepancy.

An approximation to the interaction between two "point" charges that may be of some use will be noted. In the expression for the interaction given as Eq. (6.6), we put $\mathrm{erf}(S/\sqrt{2}L) = 1$ and then carry out the integration over dS to obtain

$$W_a = (q_1q_2/R_0)\,\mathrm{erf}(R_0/\sqrt{2}L) \tag{6.12}$$

For large R_0 this has the correct asymptotic value, q_1q_2/R_0. Near the origin this approximation becomes $2^{1/2}\pi^{-1/2}q_1q_2/L$, which differs from the correct value by a factor of $2^{1/2}$. We expect the kinematical corrections to arise from effects associated with distances of the order of the parameter L. Hence, we cannot expect use of this approximation to yield precise results, but the correct qualitative behavior and fair quantitative valuation of the approximation for all distances should preclude gross errors. The shift on the $2s$ level in hydrogen may be calculated using this approximation for W to yield

$$\langle 2s|W_a|2s\rangle - \langle 2s|W_c|2s\rangle = 4^{-1}Z^4e^2a_0^{-3}L^2 \tag{6.13}$$

which is in error by a factor of 2.

We expect kinematical effects on the kinetic energy as well as on the potential energy. In Appendix B an effort is made to roughly estimate the leading effect on the former. It is found that the effect is of the same order in the parameter L as the effect on the potential energy, i.e., of order $(L/a_0)^2$, but there are factors which make the shift due to the kinetic energy about an order of magnitude smaller than the shift due to the potential energy.

Appendix A: Coordinate Systems

The probabilistic character of measurements may be envisioned in terms of repeated measurements, performed by a particular observer, of "similar" events or, alternatively, in terms of measurements of a particular event performed by numerous "equivalent" observers.

In regard to the former situation, it is necessary to define the concept of similar events. We envision an apparatus set up to repeatedly perform some experiment. The data comprise spatial and time measurements performed at the outset of the experiment, the initial conditions (which may be arbitrarily extensive in scope), and measurements performed at the conclusion of the experiment, the results. The observer performs a set of experiments. In general these experiments will be characterized by differing initial conditions and

differing results. Reiterating that we adopt a strictly classical conceptual basis and do not include probabilistic effects of quantum origin in our thinking (they are to be superimposed when dynamical calculations are made on the basis of the kinematics), the differences in the measurements associated with the various experiments are attributed to the probabilistic nature of the kinematics; i.e., in the absence of such probabilistic effects the experiment could be performed repeatedly so that every experiment would be characterized by the same initial conditions and the same results. From the set of experiments performed, we select a subset that, by chance, are characterized by the same initial conditions; i.e., the same to within whatever tolerance the experimenter prescribes. This subset of experiments is defined to be a set of similar experiments, and it is the postulate that the results will be characterized by probability distributions for spatial and time measurements.

It is perhaps intuitively helpful if one tries to envision a macroscopic situation that simulates, in a crude sense, the situation that is supposed to prevail. We may think of an observer who is making spatial measurements (in two dimensions, for conceptual simplicity) and who constructs his coordinate system on an elastic membrane. The membrane is then supposed to execute random oscillations, quite beyond the control of the observer. The coordinate system has long-term stability in the sense that the oscillations occur about a fixed equilibrium configuration and the distribution of the random oscillations has moments that do not change over long periods of time. Time measurements might be performed by a pendulum clock in which the pendulum is elastic and executes random variations in its length.

When various observers, each of whom has established his coordinate system in some arbitrary manner, perform measurements on a particular event, it is necessary that we explore the concept of equivalent observers. In order to establish the equivalence of coordinate systems it is necessary to perform a set of preliminary experiments. A repeated experiment is set up, as described in the previous situation, and, further, a standard for acceptable initial conditions is prescribed. A set of experiments is performed, but now observations are performed by a set of observers rather than by a single observer. From the set of observations made by a particular observer, the observer selects a subset of observations that have similar initial conditions and which are in accord with the standard; i.e., he selects an acceptable set of similar experiments in the sense previously defined. The various observers now compare the results of these similar, standard experiments. These results are characterized by probability distributions. It is expected that most probability distributions for a corresponding observable will differ in that the centers of the distributions differ or the moments of the distributions differ. By chance, however, there will be a subset of observers who find similar probability distributions for all the measurements that comprise the final results of the similar, standard experiments. This subset of observers constitutes the set of equivalent observers. (It need hardly be added that the foregoing procedure is not suggested as being of great practicable merit.) It is the postulate that the measurements made by the set of equivalent

observers on a single event be characterized by probability distributions for the spatial and time measurements.

A macroscopic situation which crudely simulates the situation that is supposed to prevail is a set of observers equipped with identical pendulum clocks, but the elastic pendulums execute random variations in their lengths, variations that are characterized by similar distributions but are uncorrelated one with the other. For measurement of spatial coordinates, each observer has his own elastic membrane, and the variations in the membranes are uncorrelated one with the other.

The general conceptual structure of the probabilistic character of spatial and time measurements is similar to the probabilistic character of actual laboratory measurements due to random variations in the experimental procedures. Thus, what is postulated is that there is a limit on the precision of kinematical measurements that is to be associated with the inherent structure of space and time.

The concept of a set of observers in the same Lorentz reference system is used. A particular observer may make a set of measurements of the velocity of some reference point, e.g., the origin of another observer's coordinate system. The velocity measurements will yield probability distributions for the velocity components. If the distributions center about zero velocity, the observer will conclude that the reference point is at rest relative to his coordinate system. In this manner a set of observers may ascertain that they are in the same Lorentz system.

From time to time in the discussion, the term *point* is employed, for want of a better term. The term is employed to refer to the location of some physical object and it is not intended to refer to some abstraction that would appear to be in contradiction to the conceptual basis of these considerations. Measurements of the location of the "point" are expected to yield probability distributions, in accordance with the postulates.

Appendix B: Kinematical Effect on the Kinetic Energy

The computation of kinematical effects on the potential energy does not provide guidance towards the treatment of the kinetic energy. Here, we attempt to find a prescription that will enable us to make a rough estimate of the effect on the kinetic energy of the probabilistic character of the kinematics.

It is helpful to consider the calculation of the kinetic energy in momentum space:

$$\langle \mathrm{KE} \rangle = \langle \mathbf{k}_0 | \mathrm{KE}_{\mathrm{op}} | \mathbf{k}_0 \rangle \tag{B1}$$

where $\mathbf{k}_0$ is the propagation vector. In momentum space the kinetic energy operator is simply

$$\mathrm{KE}_{\mathrm{op}} = \hbar^2 k_0^2 / 2m = m v_0^2 c^2 / 2 \tag{B2}$$

where the last form is a temporary convenience to establish a connection with kinematical considerations. The c^2 factor is introduced because, in the units adopted for the time, speeds are measured in units of the speed of light. The subscripts on k and v are to distinguish between the argument of a probability distribution (e.g., v) and the center of the distribution, the nominal value (e.g., v_0).

The viewpoint we adopt is to identify $v_0{}^2$ as the nominal value of the square of the speed, but, in accordance with the probabilistic nature of the kinematics, what heretofore was a definite value is replaced by a probability distribution for the speed. (The quantum mechanical averaging over the values of $v_0{}^2$ is deferred, in accordance with the prescription that quantum mechanical effects should be superimposed later.) Thus, a reasonable prescription is to replace $v_0{}^2$ by the second moment of the speed distribution. In order to obtain an estimate of the correction, we assume Gaussian distributions for the velocity components and for derived distributions, in which case the second moment of the speed distribution is

$$\overline{v^2} = v_0{}^2 + 2(L^2 + v_0{}^2 T^2)/H_0{}^2 \tag{B3}$$

The $v_0{}^2$ term leads to the conventional result for the kinetic energy. The other terms represent the kinematical corrections to the kinetic energy. The parameter T is expected to be of comparable magnitude to L; in fact, we have indicated an argument which equates the two parameters. Thus, the correction term in T^2 will be much smaller than the term in L^2, and the former term will no longer be considered. The leading correction to the kinetic energy may now be written as

$$D = \langle \mathrm{KE} \rangle - \langle \mathrm{KE} \rangle_c = \langle \mathbf{k}_0 | mc^2 L^2 / H_0{}^2 | \mathbf{k}_0 \rangle \tag{B4}$$

where the term $\langle \mathrm{KE} \rangle_c$ denotes the conventional result for the kinetic energy.

The interpretation to be given to the quantity H_0 is the most obscure part of the argument. The time H_0 represents the time period over which the measurement of the speed is made. One may argue that the system is in a state of well-defined energy and a relatively long time is available in which to measure the speed; then the correction term is negligible. Such a point of view seems unreasonable, and another is suggested if we write $H_0{}^2 = S_0{}^2/v_0{}^2$. The distance S_0 is the distance over which the speed measurement is made, and this is limited by the extent of the system.

With these considerations in mind we write the kinematical correction as

$$D = 2(L^2/S_0{}^2) \langle \mathbf{k}_0 | \hbar^2 k_0{}^2 / 2m | \mathbf{k}_0 \rangle = 2(L^2/S_0{}^2) \langle \mathrm{KE} \rangle_c \tag{B5}$$

For the factor $1/S_0{}^2$ we adopt the value $\langle 1/r^2 \rangle$, where r is the radial variable.

An alternative prescription might be to identify S_0 with the radial variable itself and incorporate the $1/S_0{}^2$ into the quantum mechanical operator, along with the $k_0{}^2$. The implication is that some velocity measurements must be made in very short intervals of time, which, in turn, implies velocity distributions with arbitrarily large dispersions. The conceptual structure of the

situation provides no basis for such an eventuality, and we conclude that such a prescription would be incorrect.

Unlike the corrections to the potential energy, the corrections to the kinetic energy should be comparable for the $2s$ and $2p$ levels. The energy shift between the levels is desired:

$$\text{Shift} = D(2s) - D(2p) = 2L^2\langle \text{KE}\rangle_{2s}(\langle 1/r^2\rangle_{2s} - \langle 1/r^2\rangle_{2p}) \tag{B6}$$

The result is now in a form where comparison may be made with the shift due to kinematical effects on the potential energy, which may be expressed as

$$\text{Shift(PE)} = (4Z^2L^2/a_0{}^2)\langle \text{KE}\rangle_{2s} = 16L^2\langle 1/r^2\rangle_{2s}\langle \text{KE}\rangle_{2s} \tag{B7}$$

Evaluation of the $\langle 1/r^2\rangle$ terms gives, for the shift due to kinematical effects on the kinetic energy: Shift $= (Z^2L^2/3a_0{}^2)\langle \text{KE}\rangle_{2s}$. Although kinematical corrections to both the potential and kinetic energies are of the same order in the parameter L/a_0, the correction to the kinetic energy is smaller by about an order of magnitude.

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